

Optimal Attack in the Case of No Defense

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July 17, 2001

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I. Mathematical Formulation of the Problem

This article determines an optimal allocation of weapons to targets in the case of no defense. It is assumed that the expected damage to the target is described by the "exponential" damage function (described below). Let X denote the total number of weapons, t the number of targets, and x_i the number of weapons allocated to the i -th target. The attacker's strategy is hence specified by vector

$$\underline{x}' = (x_1, x_2, \dots, x_t)$$

where each $x_i \geq 0$ and

$$\sum_{i=1}^t x_i = X,$$

where $\sum_{i=1}^t$ denotes the summation operator over the index i from $i=1$ to $i=t$.

We assume that the exponential damage function adequately describes the expected damage at each target; that is, the expected damage is

$$D_i(z_i) = P_i(1 - \exp(-\beta_i z_i))$$

where z_i is the number of weapons reaching the i -th target, P_i denotes the value of the i -th target, and β_i is the damage function parameter for the i -th target. The exponential damage function works well for point targets (e.g., geographically compact military or industrial targets or small cities). For area targets, such as large metropolitan areas, it may be desirable to use a sum of two or three exponential functions to obtain a good fit (to the damage caused to the target as a function of number of weapons deployed on the target).

Since the attacker wishes to maximize the damage, he will choose a strategy \underline{x}^* such that

$$H(\underline{x}^*) = \max(\underline{x}) H(\underline{x})$$

where $H(\underline{x})$ represents the expected damage corresponding to the strategy \underline{x} and $\max(\underline{x})$ denotes maximization with respect to \underline{x} .

II. Solution Method

The problem, hence, is to determine \underline{x}^* such that

$$H(\underline{x}^*) = \max(\underline{x}) H(\underline{x})$$

where

$$H(\underline{x}) = \sum_{(i:1,t)} D_i(x_i)$$

subject to the constraints

$$\sum_{(i:1,t)} x_i = X$$

and $x_i \geq 0$. The function $H(\underline{x})$ is additively separable over the targets, and so the Everett generalized Lagrange multiplier (GLM) technique (Reference 1) can be used to find a solution. The problem becomes one of finding x_i corresponding to

$$\max(x_i) (D_i - \lambda x_i)$$

where λ (a Lagrange multiplier) is adjusted so that the constraint

$$\sum_{(i:1,t)} x_i = X$$

is satisfied.

III. Derivation of the Solution

The optimal solution is determined by adjusting the value of the Lagrange multiplier so that the constraints are satisfied. The "beauty" of the GLM method in the case of a separable payoff function is that the problem of determining a global optimum is reduced to determining an optimal solution at each target, independently of the other targets.

Since all of the target damage functions are monotonically increasing in the number of weapons, the solution is readily determined by the bisection root-finding method. The Lagrange multiplier is positive, so a lower bound for λ is $\lambda = 0$. An upper bound is the maximum value, over the set of all targets, of the slope, $P_i\beta_i$, of the individual-target payoff functions.

IV. Solution Used in *Can America Survive?*

In the optimal attacks considered in *Can America Survive?*, the full value of a target was credited if a weapon was placed on a target. This corresponds to using a large value for β_i , say 10, for each target. In this case, targets are selected in order of their value, up to the number of weapons available. For example, if the targets are cities and the payoff function is city population, and the total attack size is $X=1,000$, then the largest 1,000 cities are attacked.

References

1. Hugh Everett, III, "Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources," Operations Research, 11:399-411, 1963.