Optimal Attack and Defense for a Number of Targets in the Case of Imperfect Interceptors

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Contents

Foreword

This article is an extraction, with some amplification and minor notational changes, of portions of the report: Caldwell, J. G., T. S. Schreiber, and S. S. Dick, Some Problems in Ballistic Missile Defense Involving Radar Attacks and Imperfect Interceptors, Report ACDA/ST-145 SR-4, Special Report 4, prepared for the US Arms Control and Disarmament Agency by the Lambda Corporation (subsidiary of General Research Corporation), McLean, Virginia, May 1969. The results presented here were derived by Dr. J. G. Caldwell. These results are presented here because they were never published in the open literature (refereed journals).

These results were also published as Appendix F of the report, Caldwell, J. G., Theater Tactical Air Warfare Methodologies: Automated Scenario Generation, Final Report Contract No. F33657-88-C-2156 prepared for the USAF Air Force Systems Command, Aeronautical System Division by Vista Research Corporation, July 1989.

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Summary

This article describes the optimal defense of a set of targets against an optimal attack, in the case of imperfect interceptors. This solution is obtained from a mathematical representation of strategic nuclear warfare as a two-sided resource-constrained optimization problem. The attacker and defender are assumed to know each other's total force sizes, and it is assumed that the attacker "moves last," i.e., the attacker allocates his weapons to targets after observing the defender's allocation of interceptors to targets. A one-on-one (or fixed salvo-to-one) firing doctrine is assumed. The attacker's goal is to maximize the total damage to the targets, and the defender's goal is to minimize this damage. It is assumed that damage on any single target can be adequately described by the "exponential" damage function defined below.

A. Mathematical Description of the Problem

1. Notation

Let us denote the total number of weapons by X and the total number of interceptors by Y. Further, let us denote the number of targets by t, and the number of weapons and interceptors allocated to the i-th target by x_i and y_i, respectively. The attacker's and defender's strategies are hence specified by vectors

$$\underline{x}' = (x_1, x_2, \ldots, x_t)$$

and

$$\underline{y}' = (y_1, y_2, \ldots, y_t)$$

where each $x_i \geq 0$, $y_i \geq 0$, and

$$\Sigma(i{:}1,t)\ x_i = X$$

and

$$\Sigma(i{:}1,t)\ y_i = Y$$

where the notation $\Sigma(i{:}1,t)$ denotes the summation operator over the index i, ranging from 1 to t.

Using the exponential damage function, the expected damage at the i-th target is

$$D_i(z_i) = P_i(1 - \exp(-\beta_i z_i))$$

where z_i is the number of weapons reaching the i-th target, P_i is the value of the i-th target, and β_i is the damage function parameter for the i-th target.

The interceptor kill probability, κ, is assumed to be the same for all targets.

The attacker is assumed to choose his strategy $\underline{x}$ after observing the strategy $\underline{y}$ selected by the defender. He will choose a strategy $\underline{x}^*(\underline{y})$ such that

$$H(\underline{x}^*(\underline{y}^*),\underline{y}) = \max(\underline{x})\ H(\underline{x},\underline{y}),$$

where $H(\underline{x},\underline{y})$ denotes the total expected damage corresponding to the strategies $\underline{x}$ and $\underline{y}$, and $\max(\underline{x})$ denotes maximization with respect to $\underline{x}$ (of the following function of $\underline{x}$, in this case $H(\underline{x},\underline{y})$). The defender hence must choose his strategy $\underline{y}^*$ such that

$$H(\underline{x}^*(\underline{y}^*),\underline{y}^*) = \min(\underline{y})\ \max(\underline{x})\ H(\underline{x},\underline{y}) ,$$

and $\min(\underline{y})$ denotes minimization with respect to $\underline{y}$.

2. An Expression for the Expected Damage

Using the exponential damage function, the expected damage on the i-th target is shown in Appendix A to be P_iQ_i where

$$Q_i = \begin{cases} 1 - \exp(-\alpha_i x_i) & , x_i \leq y_i \\ 1 - \exp(-\beta_i(x_i-y_i))\exp(-\alpha_i y_i) & , x_i > y_i , \end{cases}$$

and $\alpha_i < \beta_i$ is defined by the equation

$$\exp(-\alpha_i) = (1 - \kappa)\exp(-\beta_i) + \kappa .$$

The total expected damage is

$$H(\underline{x},\underline{y}) = \Sigma(i{:}1,t)\ H_i(x_i,y_i)$$

$$= \Sigma(i{:}1,t)\ P_iQ_i .$$

We shall denote the expected damage corresponding to the optimal solutions $\underline{x}^*$, $\underline{y}^*$ as H(X,Y).

3. Procedure for Finding the Solution

Our problem, thus, is to determine $\underline{x}^*,\underline{y}^*$ such that

$$H(\underline{x}^*,\underline{y}^*) = \min(\underline{y}) \max(\underline{x}) H(\underline{x},\underline{y})$$

where

$$H(\underline{x},\underline{y}) = \Sigma(i{:}1,t)\ P_iQ_i$$

subject to the constraints

$$\Sigma(i{:}1,t)\ x_i = X$$

$$\Sigma(i{:}1,t)\ y_i = Y$$

and $x_i \geq 0$, $y_i \geq 0$. The function $H(\underline{x},\underline{y})$ is additively separable over the targets so the Everett-Pugh double Lagrange multiplier technique (References 2 and 3) can be used to find a solution. The problem becomes one of finding x_i, y_i corresponding to

$$\min(y_i) \max(x_i) (P_iQ_i - \lambda x_i + \mu y_i)$$

where λ, μ are adjusted so that the constraints

$$\Sigma(i{:}1,t)\ x_i = X$$

and

$$\Sigma(i{:}1,t)\ y_i = Y$$

are satisfied. If there are several choices of (μ,λ) for which the constraints can be met (i.e., there are several solutions to the double Lagrange multiplier problem), the correct optimal solution to the problem is the solution corresponding the least damage.

For convenience we drop the subscript i, so that the problem is to find x, y corresponding to

$$\min(y)\ \max(x)\ (PQ - \lambda x + \mu y)\ .$$

The solution is derived in Appendix A (Technical Details). The next section will describe the results.

B. Description of the Solution

The solution to the above two-sided optimization problem is complicated to describe analytically, and can be described best in geometrical terms. Appendix A contains such a description. In this section we shall present only a qualitative description of the optimal attack and defense allocations.

Note that the mathematical approach used to solve the present problem assumes that the x_i's and y_i's are continuous variables, rather than integers. This approximation introduces a negligible error whenever the number of interceptors and weapons allocated to targets are zero or large enough that the fractional parts are insignificant.

The optimal solution can conveniently be described by observing how the weapon and interceptor allocations to targets change for a fixed total defense level, Y, as we very the total offense level, X, from zero to a very large number. In all situations, weapons are allocated to targets so that the marginal payoff per weapon is the same on all targets. Note that for a specified X, there may be some targets that, even if undefended, yield so little payoff to the attacker that they are not attacked. Such targets are not defended, either. For a specified total defense level, Y, there are two total attack levels, $X_{min}(Y)$ and $X_{max}(Y)$, that are of special significance. We shall describe the nature of the solution in terms of these levels.

1. Low Attack Level ($X < X_{min}(Y)$)

When the attack level is less than the value $X_{min}(Y)$, every attacked target is defended by a number of interceptors that exceeds the number of weapons assigned to it. Thus damage to targets can occur only through "leakage", i.e., failure of the interceptors to kill all incoming weapons. The defense allocation is arbitrary, so long as there are more interceptors than weapons on each attacked target. The attack allocation for specified X is unique. The total payoff to the attacker is an increasing concave ("diminishing returns") function of the total attack level, X.

2. Intermediate Attack Level ($X_{min}(Y) < X < X_{max}(Y)$)

For all attack levels lying between the values $X_{min}(Y)$ and $X_{max}(Y)$, the defense is unique. Such a defense is called "stable." In this region of attack levels, the attacker can choose between two different attack levels on each target. The lower level may be zero on some targets. The attacker must attack each target at either the lower level or the higher level, and can choose to attack at any combination of these permissible levels that meets his total attack level, X. The payoff is a linear function of the total attack level in this region.

The situation in the case of the low and intermediate attack levels is referred to as "defense dominant".

3. High Attack Level ($X > X_{max}(Y)$)

As the total number of weapons is increased above the level $X_{max}(Y)$, the defender must change his allocation. He does so by defending fewer targets, but defending the defended targets at a higher level. As the attack size increases, eventually only one target would be attacked. Also, for a sufficiently large attack size all targets would be attacked. Since the defense allocation changes as the attack size changes in this region, the defense is called "unstable". Both the attack and defense allocations corresponding to specified X are unique.

The situation in the case of a high attack level is called "attack dominant".

4. Qualification Concerning the Lagrangian Solution

The solutions to the present problem are derived by the method of generalized Lagrange multipliers. Using this technique, we can find a solution for every total attack level $X < X_{min}(Y)$. Over the range $X_{min}(Y) \leq X \leq X_{max}(Y)$, however, there are only a finite number of Lagrangian solutions, corresponding to all possible combinations of low level or high level attacks on each attacked target. For a large number of weapons, interceptors, and targets, there is a very large number of such solutions, and there would be a Lagrangian solution corresponding to a total attack size quite close to any specified total attack level. Hence in practice we need only concern ourselves with the Lagrangian solutions. (Solutions corresponding to other total attack levels differ very little from Lagrangian solutions corresponding to nearby total attack levels. These other solutions are less "efficient" than the Lagrangian solutions, since the payoff function for them lies below the marginal payoff for the additional weapons used (beyond those required by the closest Lagrangian solution using fewer weapons) is less than the total marginal payoff achieved by going to the next higher Lagrangian solution. This latter marginal payoff is defined by the slope of linear payoff function defined by the Lagrangian solutions.)

Over the range $X > X_{max}(Y)$, there can be found a Lagrangian solution for any specified total attack level, X, but not for every specified total defense level, Y. The reason for this difficulty lies with the fact that, in the Lagrangian solution, a target can be defended at only two levels, one of which is zero (no defense). (The situation is analogous to the intermediate attack situation in which targets could be attacked at only two levels.) As before, this problem is not of practical significance, since we can always find a Lagrangian solution corresponding to a total defense level close to the specified total defense level.

C. Comparison of Solution in the Case of Imperfect Interceptors to That for Perfect Interceptors

It is of interest to compare how the nature of the solution to the allocation problem in the case of imperfect interceptors differs from the solution in the case of perfect interceptors. Two questions of interest are:

1. For what range of the kill probability, κ, are the allocations and total damage essentially the same as for the perfect interceptor case, $\kappa = 1$?

2. For specified interceptor kill probability, κ, and total defense level, Y, is the solution to the problem "similar" (in allocations and total damage) to the perfect interceptor problem with total defense level κY? (Note that κY is the expected number of interceptors that do not fail, and represents a possible "effective" number of perfect interceptors.)

The questions were answered by examining the nature of the solution for a "reasonable" set of hypothetical targets, described in Table 1.

Table 1. Hypothetical Target Set

Target Number	Target Value (Population in millions)	Exponential Damage Function Parameter (β)
1	10	.03
2	6	.04
3	4	.05
4	3	.06
5	2	.08
6	1.5	.10

7	1.0	.20
8	1.0	.30
9	1.0	.40
10	.5	.50

The total expected damage, H(X,Y), corresponding to the optimal attack and defense allocations is plotted in Figures 1 and 2 as a function of total attack level, X, for fixed total defense level, Y.

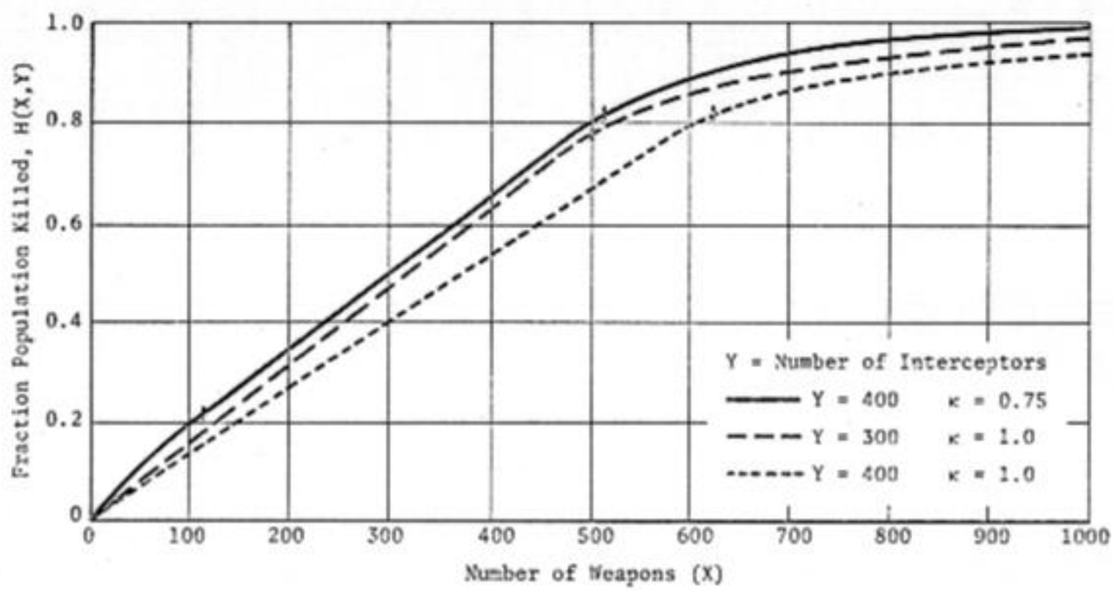

Figure 1. Relationship of Payoff for Optimal Solution to Interceptor Kill Probability - Low Defense Level

Figure 1 corresponds to Y_1 = 400, and Figure 2 corresponds to Y_2 = 800. On each figure are three curves:

1. $H(X,Y_i)$ for κ = 1
2. $H(X,Y_i)$ for κ = .75

3. $H(X,.75Y_i)$ for $\kappa = 1$.

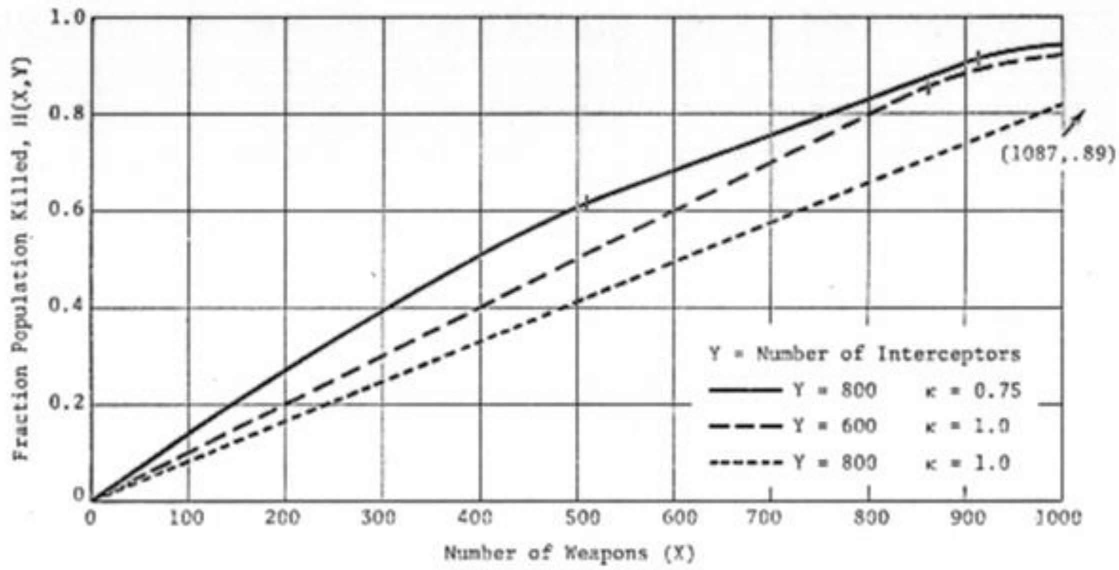

Figure 2. Relationship of Payoff for Optimal Solution to Interceptor Kill Probability - High Defense Level

The figures indicate first that there are substantial differences in total damage between the $\kappa = 1$ and $\kappa = .75$ cases, and that the imperfect interceptor case is not always closely approximated by the perfect interceptor case for total defense level κY. However, the difference between these latter two cases is not so great as to preclude using the "equivalent perfect interceptor" approximation for some purposes.

Appendix A. Technical Details

I. Introduction

This appendix derives the optimal defense of a set of targets against an optimal attack in the case of imperfect interceptors. It is assumed that the defender has a total number, Y, of interceptors, and that the attacker has a total number, X, of weapons. Let us denote the number of targets by t, and the number of weapons and interceptors allocated to the i-th target by x_i and y_i, respectively. The attacker's and defender's strategies are hence specified by vectors

$$\underline{x}' = (x_1, x_2, \ldots, x_t)$$

and

$$\underline{y}' = (y_1, y_2, \ldots, y_t)$$

where each $x_i \geq 0$, $y_i \geq 0$, and

$$\Sigma(i{:}1,t)\ x_i = X$$

and

$$\Sigma(i{:}1,t)\ y_i = Y\ .$$

We assume that the exponential damage function adequately describes the expected damage at each target; that is, the expected damage is

$$D_i(z_i) = P_i(1 - \exp(-\beta_i z_i))$$

where z_i is the number of weapons reaching the i-th target, P_i denotes the value of the i-th target, and β_i is the damage function parameter for the i-th target.

We assume that the attacker "moves last," i.e., he chooses his strategy, $\underline{x}$, after observing the strategy $\underline{y}$ selected by the defender. He will choose a strategy $\underline{x}^*(\underline{y})$ such that

$$H(\underline{x}^*(\underline{y}),\underline{y}) = \max(\underline{x})\ H(\underline{x},\underline{y})$$

where $H(\underline{x},\underline{y})$ represents the expected damage corresponding to the strategies $\underline{x}$ and $\underline{y}$. The defender, then, should choose his strategy $\underline{y}^*$ such that

$$H(\underline{x}^*(\underline{y}^*),\underline{y}^*) = \min(\underline{y})\ \max(\underline{x})\ H(\underline{x},\underline{y}) = U\ .$$

The value U is the upper pure value of the game in which the attacker and defender move simultaneously. Since $\underline{x}^*$ and $\underline{y}^*$ are pure strategies, there is no reason why U should equal the lower pure value of such a game, defined by

$$L = \max(\underline{x})\ \min(\underline{y})\ H(\underline{x},\underline{y})\ .$$

Both the attacker and defender are assumed to know each other's force size and κ.

We assume that the interceptor kill probability, κ, is the same for all targets.

II. An Expression for the Expected Damage

The expected damage on the i-th target if u weapons get through is given by

$$D_i(u) = P_i(1 - \exp(-\beta_i u)) .$$

The probability that u weapons get through, given that there are x_i weapons and y_i interceptors, both integers, is

$$P_{x_iy_i}(u) = \begin{cases} A \text{ if } 0 \le u \le x_i \text{ and } x_i \le y_i \\ 0 \text{ if } 0 \le u < x_i - y_i \text{ and } x_i > y_i \\ B \text{ if } x_i - y_i \le u \le x_i \text{ and } x_i > y_i \end{cases}$$

where

$$A = C(x_i,u)(1 - \kappa)^u \kappa^{xi - u}$$

and

$$B = C(y_i,(u - (x_i - y_i)))\ (1-\kappa)^{u - (xi - yi)} \kappa^{xi - u} ,$$

and C(n,e) denotes the combination function (i.e., the number of combinations of n things taken e at a time). (Note: The word processing program used to display this article cannot place subscripts or superscripts on subscripts or superscripts; the exponents xi and yi in the preceding expressions (and in similar expressions that follow) should be x_i and y_i, respectively.)

The expected damage on the i-th target is hence

$H_i(x_i,y_i) = \Sigma(u:0,x_i)$ (Expected damage given that u weapons get through) (Probability that u weapons get through)

$$= \Sigma(u:0,x_i)\ P_i(1 - \exp(-\beta_i u))P_{x_i y_i}(u)$$

$$= P_i\Sigma(u:0,x_i)\ (1 - \exp(-\beta_i u))P_{x_i y_i}(u)$$

$$= P_i Q_i\ ,$$

where

$$Q_i = \begin{cases} A, & x_i \le y_i \\ B, & x_i > y_i \end{cases}$$

where

$$A = 1 - ((1-\kappa)\exp(-\beta_i)+\kappa)^{x_i}$$

and

$$B = 1 - \exp(-\beta_i(x_i - y_i))\ ((1 - \kappa)\exp(-\beta_i) + \kappa)^{y_i}\ .$$

We shall use this expression for all x_i, y_i, not just for integral values. We shall simplify the expression for Q_i by defining

$$\exp(-\alpha_i) = (1 - \kappa)\exp(-\beta_i) + \kappa$$

where $\alpha_i \le \beta_i$ since $\kappa \le 1$. Hence we have

$$Q_i = \quad 1 - \exp(-\alpha_i x_i) \qquad , x_i \le y_i$$

$$1 - \exp(-\beta_i(x_i - y_i)) \exp(-\alpha_i y_i) \quad , x_i > y_i \, .$$

The total expected damage is

$$H(\underline{x},\underline{y}) = \Sigma(i{:}1,t)\ H_i(x_i,y_i)$$

$$= \Sigma(i{:}1,t)\ P_i Q_i \, .$$

We shall denote the expected damage corresponding to the optimal solutions $\underline{x}$*, $\underline{y}$* as H(X,Y).

III. Procedure for Finding the Solution

Our problem, thus, is to determine $\underline{x}$*, $\underline{y}$* such that

$$H(\underline{x}^*,\underline{y}^*) = \min(\underline{y})\ \max(\underline{x})\ H(x,y)$$

where

$$H(\underline{x},\underline{y}) = \Sigma(i{:}1,t)\ P_i Q_i$$

subject to the constraints

$$\Sigma(i{:}1,t)\ x_i = X$$

$$\Sigma(i{:}1,t)\ y_i = Y$$

and $x_i \geq 0$, $y_i \geq 0$. Since the function H($\underline{x}$,$\underline{y}$) is not concave, the Kuhn-Tucker theorem is of no assistance in computing the solution. The function H($\underline{x}$,$\underline{y}$) is additively separable over the targets, however, and so the Everett-Pugh double Lagrange multiplier technique (References 2 and 3) can be used to find a solution. The problem becomes one of finding x_i, y_i corresponding to

$$\min(y_i) \max(x_i) (P_iQ_i - \lambda x_i + \mu y_i)$$

where λ, μ are adjusted so that the constraints

$$\Sigma(i{:}1,t)\ x_i = X$$

and

$$\Sigma(i{:}1,t)\ y_i = Y$$

are satisfied. If there are several choices of (λ,μ) for which the constraints can be met, (i.e., there are several solutions to the double Lagrange multiplier problem), the correct optimal solution to the problem is the solution corresponding to the least damage. We shall for convenience drop the subscript i; that is, the problem is to find x, y corresponding to

$$\min(y) \max(x) (PQ - \lambda x + \mu y) .$$

The values of the Lagrange multipliers are adjusted so that the resource constraints are met.

The significant feature of the generalized Lagrange multiplier (GLM) method is that, because the damage function is additively separable (i.e., the total damage is the sum of the damage on each target), the solution of the global optimization problem may be found by simply solving, independently, an unconstrained optimization problem for each target (and then adjusting the multipliers so that the constraints are obeyed. Another reason (apart from additive separability) why the GLM method is used in this problem is that the damage function is not convex (so that the Kuhn-Tucker and other standard optimization methods cannot be applied). (In addition to not requiring convexity of the objective function, the GLM method also does not require differentiability or continuity or linearity of the objective function.)

The only complexity that arises in the present problem is that the two-sided GLM procedure may produce a multiplicity of solutions for specified values of the Lagrange multipliers, some of which may not correspond to optimal solutions. It is hence necessary to prove that the optimal solution is contained within the solution set, and to select it from the non-optimal solutions. (This difficulty does not occur with the one-sided GLM method.)

IV. Derivation of the Solution

The damage function H(x,y) = PQ may be illustrated as in Figure A-1.

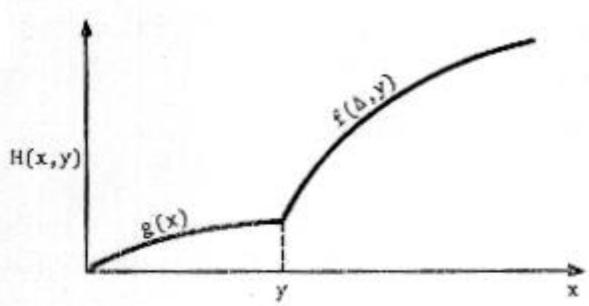

Figure A-1. Exponential Damage Function: Imperfect Interceptors

Let us denote

$$g(x) = P(1 - \exp(-\alpha x))$$

and

$$f(\Delta,y) = P(1 - \exp(-\beta\Delta - \alpha y))$$

where $\Delta = x - y > 0$. Then we may write

$$PQ = H(x,y) = \begin{cases} g(x) & , x \leq y \\ f(\Delta,y) & , x > y . \end{cases}$$

We distinguish three cases:

Case 1: $g'(0) \leq \lambda$, $f'(0,0) > \lambda$

Case 2: $g'(0) > \lambda$ (which implies $f'(0,0) > \lambda$ since $\alpha < \beta$)

and

Case 3: $f'(0,0) \leq \lambda$ (which implies $g'(0) \leq \lambda$ since $\alpha < \beta$) ,

where the derivative of $f(\Delta,y)$ is with respect to Δ.

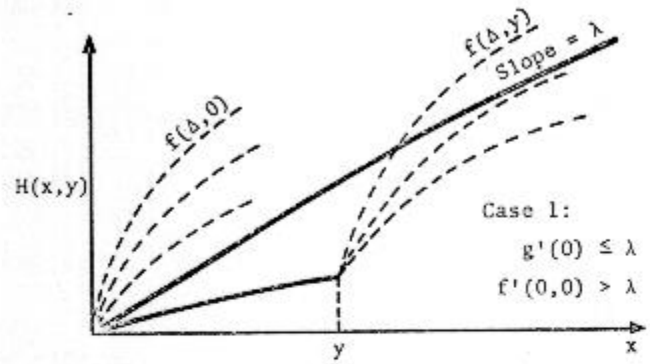

Figure A-2. Damage Function in Case 1

These cases are depicted in Figures A-2, A-3, and A-4.

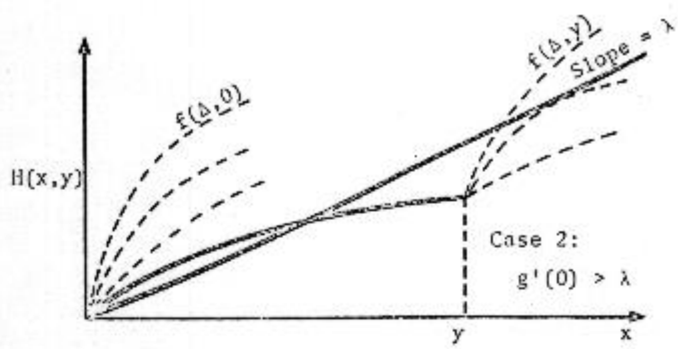

Figure A-3. Damage Function in Case 2

(The parenthetic remarks follow since:

$g(x) = P(1 - \exp(-\alpha x)$
$g'(x) = P\alpha \exp(-\alpha x)$
$g'(0) = P\alpha$
$f(\Delta,y) = P(1 - \exp (-\beta\Delta - \alpha y)$
$f'(\Delta,y) = P\beta \exp(-\beta\Delta - \alpha y)$
$f'(0,0) = P\beta$

$\alpha < \beta$ therefore $P\alpha < P\beta$, and hence $g'(0) < f'(0,0)$.)

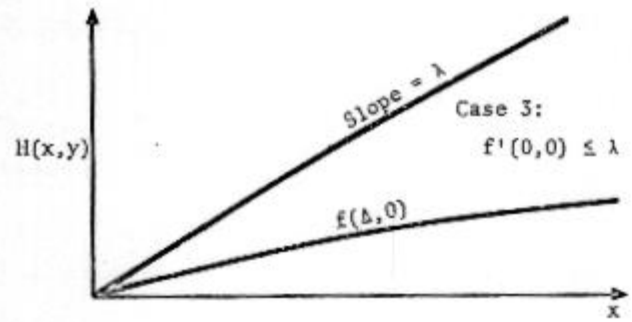

Figure A-4. Damage Function in Case 3

A. Solutions for Case 1 and Case 3

Now Case 1 is a straightforward generalization of the problem solved by Goodrich (pp. 12 - 19 of Reference 4). Goodrich solved this problem in the case of perfect interceptors ($\kappa = 1$), as depicted in Figure A-5.

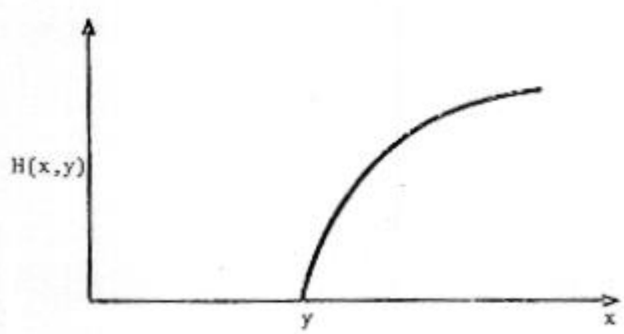

Figure A-5. Damage Function in the Case of Perfect Interceptors

The solution in Case 1 is as follows. For specified y, consider the line passing through the origin and tangent to the damage curve (see Figure A-6). (Note: The Figure A-6 does not clearly show that the damage curve has slope lower than the tangent line at the origin.)

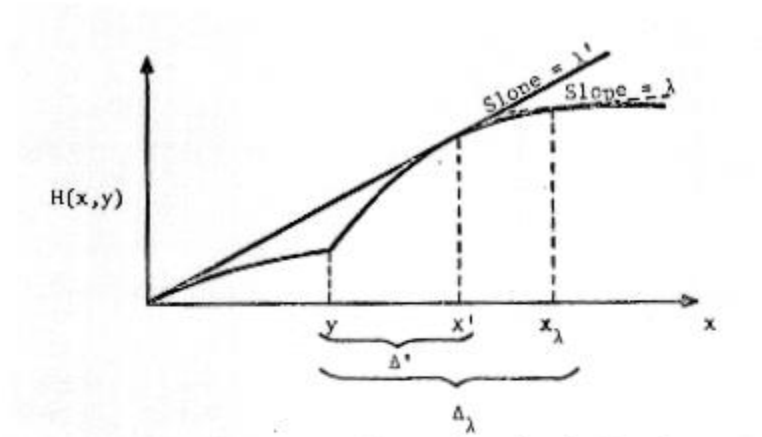

Figure A-6. Damage Function in Case 1

Let us denote the slope of this line by λ'(y), the value of x at the point of tangency by x', and denote Δ'(y) = Δ' = x' - y. Let us denote the value of x corresponding to slope λ by x_λ, and denote $\Delta_\lambda(y) = \Delta_y = x_\lambda - y$. It is easy to show (see Goodrich) that the value of x that maximizes the Lagrangian function

$$L(x,y) = H(x,y) - \lambda x + \mu y$$

is

$$x^* = \begin{cases} 0 & \text{if } \lambda > \lambda'(y) = \lambda' \\ x_\lambda & \text{if } \lambda < \lambda'(y) = \lambda' \end{cases}$$

(either value, 0 or x_λ, may be chosen if $\lambda = \lambda'$). Denoting $\Delta'(y) = \Delta' = x' - y$, we have that

$$f'(\Delta',y) == \lambda' = f(\Delta',y)/(\Delta' + y) ,$$

(where the differentiation is with respect to Δ and the double equal sign, ==, denotes "is identically equal to") and we may rewrite the optimal x-solution as

$$x^* = \begin{cases} 0 & \text{if } \lambda > f(\Delta',y)/(\Delta' + y) \\ x_\lambda = \Delta_\lambda + y & \text{if } \lambda < f(\Delta',y)/(\Delta' + y) . \end{cases}$$

Note that if $f'(0,0) < \lambda$ (Case 3), then $x^* = 0$, for then (since $f'(\Delta,y) < f'(0,0) < \lambda$) there is no point x such that $f'(\Delta,y) = \lambda$, regardless of the value of y). (This follows by the following:

$$f'(\Delta,y) = P\beta \exp(-\beta\Delta - \alpha y)$$
$$f'(0,0) =-P\beta$$
$$f'(\Delta,y) = P\beta \exp(-\beta\Delta - \alpha y) < f'(0,0) \text{ (since } \exp(-\beta\Delta - \alpha y) < 1)$$

so if $f'(0,0) < \lambda$ then $f'(\Delta,y) < \lambda$ for all x,y, so that there are no roots if $f'(0,0) < \lambda$.)

Substituting the above solution x^* into $L(x,y)$, we obtain

$$L(x^*,y) = \begin{cases} \mu y & \text{if } \lambda > f(\Delta',y)/(\Delta'+y) \\ f(\Delta_\lambda,y) - \lambda(\Delta_\lambda+y) + \mu y & \text{if } \lambda < f(\Delta',y)/(\Delta'+y) . \end{cases}$$

Note that we have substituted $\Delta_\lambda + y$ for x_λ in the above, because we wish to represent y explicitly. We wish to choose y to minimize $L(x^*,y)$.

It is interesting, but not necessary for the solution, to note that $f(\Delta_\lambda,y)$ depends only on λ, in the case of the exponential damage function. We will now demonstrate this. Now Δ_λ is defined by the equation

$$f'(\Delta_\lambda,y) = \lambda$$

where the differentiation of $f(\Delta,y)$ is with respect to Δ. Note that Δ_λ depends on y. Since

$$f(\Delta,y) = P(1 - \exp(-\beta\Delta - \alpha y)) ,$$

we have

$$f'(\Delta,y) = P\beta\exp(-\beta\Delta - \alpha y) ,$$

so that

$$P\beta\exp(-\beta\Delta_\lambda - \alpha y) = \lambda ,$$

so that

$$f(\Delta_\lambda,y) = P(1 - \lambda/P\beta) ,$$

which depends only on λ. Hence we could write

$$f(\Delta_\lambda,y) = f(\lambda)$$

in the case of the exponential damage function.

This result is interesting, since if implies that the damage is the attacker's (heavier) optimal attack corresponding to the value λ is the same (for the exponential damage function), whether the defender chooses to defend the target or not (since the height of the damage function is the same ($f(\Delta_\lambda,0) = f(\Delta_\lambda,y)$) in either case). (If, in the optimal attack against a defended target, the attacker has two attack options (e.g., 0 and x_λ), the no-defense damage (optimal attack) is the same as the damage at the <u>heavier</u> optimal attack level against the defended target.) We observe from the expression for $f(\lambda)$ that the fraction surviving the optimal attack is $\lambda/P\beta$.)

Continuing with the (general) solution, we observe that we can write the Lagrangian function as

$$L(x^*,y) = \begin{cases} \mu y & \text{if } \lambda > f(\Delta',y)/(\Delta' + y) \\ f(\lambda) - \lambda(\Delta_\lambda + y) + \mu y & \text{if } \lambda < f(\Delta',y)/(\Delta' + y) \end{cases}$$

$$= \begin{cases} \mu y & \text{if } y > f(\Delta',y)/\lambda - \Delta' \\ f(\lambda) - \lambda(\Delta_\lambda + y) + \mu y & \text{if } y < f(\Delta',y)/\lambda - \Delta' . \end{cases}$$

Now the set

$$S = \{y \mid y > f(\Delta'(y),y)/\lambda - \Delta'(y)\}$$

$$= \{y \mid \lambda > f(x'+y,y)/x'\}$$

$$= \{y \mid \lambda > \lambda'(y)\} .$$

Now $\lambda'(y) = f(x'+y,y)/x'$ is a monotonic increasing function of y, and so

$$S = \{y \mid y > y'\}$$

where y' is the unique solution to the equation

$$y = f(\Delta'(y),y)/\lambda - \Delta'(y) .$$

Hence the Lagrangian becomes

$$L(x^*,y) = \begin{cases} \mu y & \text{if } y > y' \\ f(\Delta_\lambda,y) - \lambda(\Delta_\lambda+Y) + \mu y & \text{if } y < y' . \end{cases}$$

(For the exponential damage function, $f(\Delta_\lambda(y),y) = f(\lambda)$, and y' is that y such that

$$f(\Delta'_\lambda(y),y)/(y + \Delta'_\lambda(y)) = \lambda$$

$$\text{i.e., } f(\lambda)/(y' + \Delta'_\lambda(y')) = \lambda$$

$$f(\lambda)/ \lambda = y' + \Delta'_\lambda (y')$$

$$y' = f(\lambda)/\lambda - \Delta'_\lambda(y') .)$$

Since L(x*,y) is hence a piecewise linear function of y, the value of y that minimizes L(x*,y) occurs either as y = 0 or at the solution y = y' to the equation

$$y = f(\Delta'(y),y)/\lambda - \Delta'(y) .$$

Substituting these values into L(x*,y), we obtain

$$L(x^*,0) = f(\Delta_\lambda(0),0) - \lambda\Delta_\lambda(0)$$

and

$$L(x^*,y') = \mu y'$$

$$= (\mu/\lambda)(f(\Delta'(y'),y') - \lambda\Delta'(y')) \ .$$

To minimize $L(x^*,y)$, we hence choose

$$Y = \begin{cases} 0 & \text{if } \mu > \lambda c \\ y' & \text{if } \mu < \lambda c \end{cases}$$

where

$$c = (f(\lambda) - \lambda\Delta_\lambda(0))/(f(\lambda) - \lambda\Delta_\lambda(y')) \ ,$$

and either value if $\mu = \lambda c$. This corresponds to the solution obtained by Goodrich (where $c = 1$ in the case of perfect interceptors). (The solution represents a "balanced defense", since the per-weapon payoff to the attacker is the same for all targets.)

B. Solution for Case 2

We now consider Case 2, in which $g'(0) > \lambda$. (This implies $f'(0,0) > \lambda$. Observe also that $f'(0,x) > g'(x)$.) For specified y, consider the line tangent to the damage function in two places, as shown in Figure A-7.

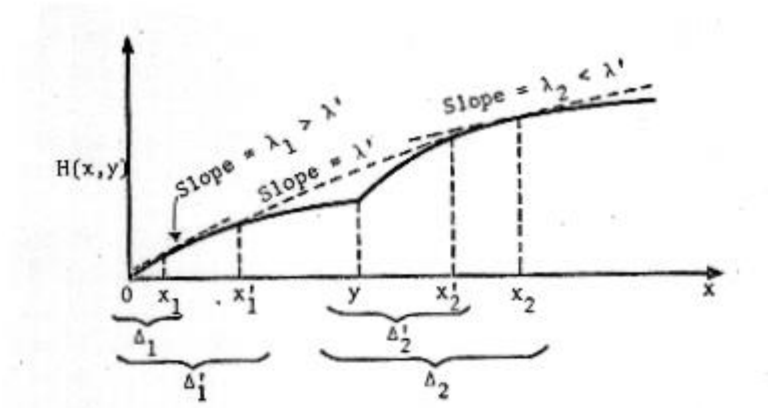

Figure A-7. Damage Function in Case 2

We denote the slope of this line by λ', and the values of x at the two points of tangency by x_1' and x_2'. We define x_1, x_2, Δ_1, Δ_2, Δ_1', Δ_2' similarly, as shown in Figure A-7. Since $f'(0,x) > g'(x)$, we have $x_1' < y < x_2'$. Since we are assuming that $g'(0) > \lambda$, the value of x that maximizes the Lagrangian function

$$L(x,y) = H(x,y) - \lambda x + \mu y$$

cannot occur at x = 0. Rather, the optimal solution is

$$x^* = \begin{cases} x_1 & \text{if } \lambda > \lambda' \\ x_2 & \text{if } \lambda < \lambda' \end{cases}$$

(either value if $\lambda = \lambda'$). That this is the optimal solution follows from the generalizated theory of Lagrange multipliers (Reference 1). Defining $\Delta_1' = x_1'$ and $\Delta_2' - x_2' - y$, we have

$$\lambda' == f'(\Delta_2',y) = (f(\Delta_2',y) - g(\Delta_1'))/(\Delta_2' + y - \Delta_1')$$

(where the differentiation is with respect to Δ_2') so that we may write

$$x^* = \begin{cases} x_1 = \Delta_1 & \text{if } \lambda > (f(\Delta_2',y) - g(\Delta_1'))/(\Delta_2' + y - \Delta_1') \\ x_2 = \Delta_2 + y & \text{if } \lambda < (f(\Delta_2',y) - g(\Delta_1'))/(\Delta_2' + y - \Delta_1') . \end{cases}$$

As in Case 1, we observe that $f(\Delta_2,y)$ depends only on λ in the case of the exponential damage function. The Lagrangian function hence becomes

$$L(x^*,y) = \begin{cases} g(\Delta_1) - \lambda\Delta_1 + \mu y & \text{if } \lambda > (f(\lambda') - g(\Delta_1'))/(\Delta_2' + y - \Delta_1') \\ f(\lambda)-\lambda(\Delta_2 + y)+\mu y & \text{if } \lambda < (f(\lambda') - g(\Delta_1'))/(\Delta_2' + y - \Delta_1') \end{cases}$$

$$= \begin{cases} g(\Delta_1) - \lambda\Delta_1 + \mu y & \text{if } y > y' \\ f(\lambda) - \lambda(\Delta_2 + y) + \mu y & \text{if } y < y' \end{cases}$$

where, as in Case 1, we substituted $\Delta_2 + y$ for x_2 and where y' is the solution to the equation

$$y = (f(\lambda) - g(\Delta_{1\lambda}))/\lambda - (\Delta_{2\lambda}(y) - \Delta_{1\lambda}) .$$

Note that $L(x^*,y)$ is continuous at $y = y'$. Since $L(x^*,y)$ is piecewise linear in y, the minimum occurs either at $y = 0$ or at the solution $y = y'$ to

$$y = (f(\lambda) - g(\Delta_{1\lambda}))/\lambda - (\Delta_{2\lambda}(y) - \Delta_{1\lambda}) .$$

(It is easy to show that the function $L(x^*,y)$ is always positive for y positive.)

Calculating the value of $L(x^*,y)$ for these two values, we have

$$L(x^*,0) = f(\lambda) - \lambda\Delta_2(0)$$

and

$$L(x^*,y) = g(\Delta_1) - \lambda\Delta_1 + \mu y'$$

$$= g(\Delta_1) - \lambda\Delta_1 + \mu((f(\lambda) - g(\Delta_1))/\lambda - (\Delta_2(y') - \Delta_1)).$$

To minimize $L(x^*,y)$ we hence choose y to be

$$y^* = \begin{cases} 0 & \text{if } \mu > \lambda c \\ y' & \text{if } \mu < \lambda c \end{cases}$$

where

$$c = [(f(\lambda) - g(\Delta_1))/\lambda - (\Delta_2(0) - \Delta_1)] / [(f(\lambda) - g(\Delta_1))/\lambda - (\Delta_2(y') - \Delta_1)],$$

and either value if $\mu = \lambda c$.

C. Summary of Lagrangian Solutions

Let us denote the attacker's optimal solution corresponding to y^* by

$$x^* = x^*(y^*)$$

$$= \begin{cases} 0 \text{ or } x' & \text{if } y^* = y' \end{cases}$$

x'' if $y^* = 0$

for Case 1 targets, and

$$= \begin{cases} x_1' \text{ or } x_2' & \text{if } y^* = y' \\ x'' & \text{if } y^* = 0 \end{cases}$$

for Case 2 targets.

Both x* and y* are zero for Case 3 targets; that is, the only targets worth attacking are those worth defending. Note that, since $f'(x,0) > g'(x)$ for small x, $x'' > x_1'$ for small x_1'. Further, since $f'(x - y,y) < f'(x - y,0)$, we have $x'' > x_2' - y'$.

An unfortunate aspect of the solutions obtained by the double Lagrange multiplier method is that, for specified resource levels (X and Y), there may be several solutions, corresponding to different values of the Lagrange multipliers (μ and λ). While all such solutions are optimal for the attacker, only one of them is also optimal for the defender. We shall refer to this latter solution as the "correct" optimal solution. (The others are referred to as "spurious".) The next three sections well describe the Lagrangian solutions in further detail, and which of them are optimal for both the attacker <u>and</u> the defender simultaneously (i.e., which combinations correspond to "correct" optimal solutions.)

V. Characteristics of the Attacker's Optimal Solutions

We shall now observe some properties of the attacker's optimal solutions. The salient characteristic of the solution is that the attacker attacks each target at a level for which the slope of damage curve is the same (λ). It is noted that the defender either allocates no interceptors to a target, or y' interceptors to a target, where y' is the level such that the line of slope λ tangent to the damage curve at the attack level x > y' either passes through the origin (in Case (1) in which g'(0) < λ), or is also tangent to the damage curve at the attack level x < y' (in Case (2) in which g'(0) > λ).

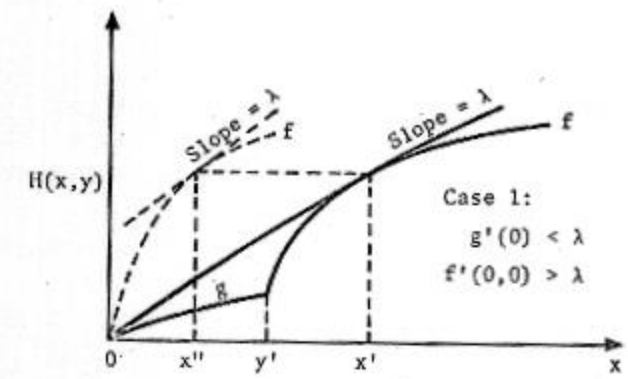

Figure A-8. Lagrangian Solutions (0,y')(x',y') or (x'',0) if g'(0) < λ and f'(0,0) > λ

This situation is illustrated in Figures A-8, A-9, and A-10.

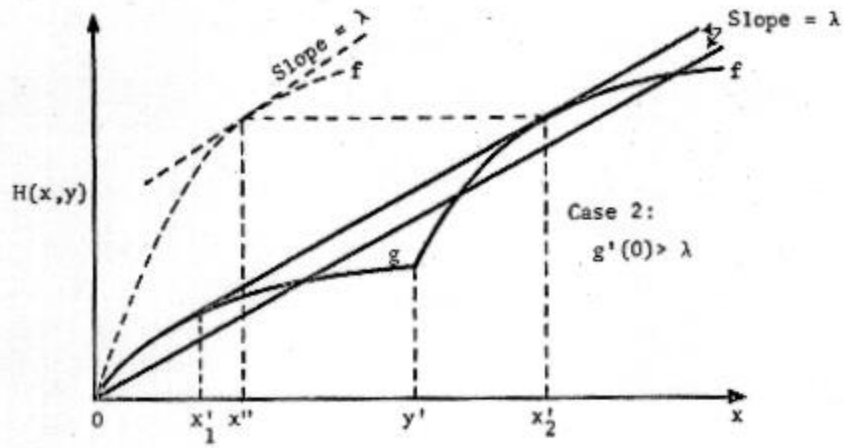

Figure A-9. Lagrangian Solutions (x_1',y') or (x_2',y') or (x",0) if g'(0)> λ

(Concerning Figure A-9: Since f'(0,x) > g'(x), there must exist Δ > 0 and y > 0 such that f'(Δ,y) = λ in this case. Note that x" > x_1' holds only if x_1' is small.)

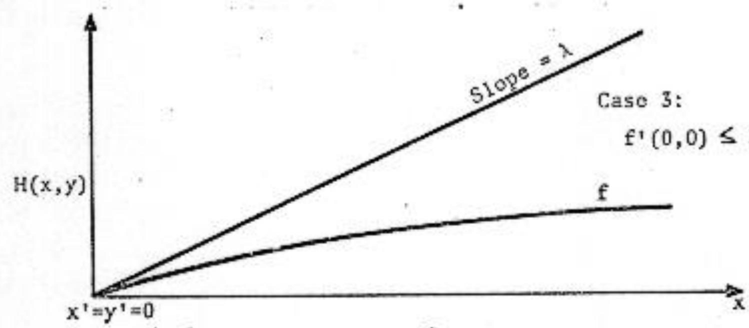

Figure A-10. Lagrangian Solution (x' = y' = 0) if f'(0,0)≤ λ

(Concerning Figure A-10: Since f'(0,0) > g'(x), there is no x such that g'(x) = λ.)

Note that the attacker chooses his solution after observing at each target whether the defender defends at level 0 or y'. For reasons that will be explained later, the attacker can attack some targets at the low level (0 for Case 1 targets and x_1' for Case 2 targets) only if the defender is defending all Case 1 and Case 2 targets at the higher level (y').

In summary, the optimal solution for the attacker has the following properties:

1. The <u>marginal</u> payoff to the attacker on all targets attacked is λ.

2. On all defended "Case 1" targets, the <u>ratio</u> of damage to number of weapons is λ; on all defended "Case 2" targets, the ratio of damage to number of weapons is greater than λ, and this ratio is greater for the solution x_1' than the solution x_2'; on all undefended Case 1 or Case 2 targets, the ratio is greater than λ.

VI. Characteristics of the Defender's Optimal Solutions

If all Case 1 or Case 2 targets are defended, the total number,

$$Y_{max}(\lambda) == \Sigma\, y_i'$$

where the symbol Σ denotes summation; if all Case 1 and Case 2 targets are defended, and attacked at the highest optimal level, the total number of weapons used is

$$X_{max}(Y_{max}(\lambda)) == X_{max}(\lambda) = \Sigma\, x_{2i}' \;;$$

if all Case 1 and Case 2 targets are defended, and attacked at the lowest optimal level, the total number of weapons used is

$$X_{min}(Y_{max}(\lambda)) == X_{min}(\lambda) = \Sigma\, x_{1i}' \;;$$

if all Case 1 and Case 2 targets are not defended, but are attacked at the optimal level, the total number of weapons used is

$$X(Y,\lambda)\big|_{Y=0} = X(0,\lambda) == \Sigma\, x_{3i}' \;;$$

where we define

$$x_{1i}' = \begin{cases} 0 \text{ on Case 1 targets} \\ x_1' \text{ on Case 2 targets} \\ 0 \text{ on Case 3 targets} \end{cases}$$

(corresponding to minimal attack, maximal defense);

$$x_{2i}' = \begin{cases} x' \text{ on Case 1 targets} \\ x_2' \text{ on Case 2 targets} \\ 0 \text{ on Case 3 targets} \end{cases}$$

(corresponding to maximal attack, maximal defense); and

$$x_{3i}' = \begin{cases} x'' \text{ on Case 1 or Case 2 targets} \\ 0 \text{ on Case 3 targets} \end{cases}$$

(corresponding to an attack against no defense).

The following characteristics of the defender's solution are noted:

1. For the complete defense, $Y = Y_{max}(\lambda)$, the attacker can choose not to attack any (or all) of the Case 1 targets or to attack any of the Case 2 targets at the lower level, x_1', and the resulting deployments (attack and defense) are optimal at the new level x,

$$X_{min}(\lambda) \leq X \leq X_{max}(\lambda) .$$

This characteristic results from the defender's having chosen the defense so that the tangent at slope λ passes through the origin (Case 1) or through both points of tangency (Case 2). The attacker is thus indifferent to attacking at the level 0 or the level x' (Case 1), or to attacking at the level x_1' or x_2' (Case 2). The damage increases linearly (with slope λ) for all X, $X_{min}(\lambda) \leq X \leq X_{max}(\lambda)$ (this will be proved later).

2. If the defender chooses not to defend some Case 1 or Case 2 targets, and the attacker <u>also</u> chooses not to attack some defended Case 1 targets or to attack some defended Case 2 targets at the lower level, then the resulting deployment is <u>not optimal</u> for the given resource levels, X and Y, even though the deployment would be the optimal one for the given values of λ and μ, if λ and μ represented true marginal costs of weapons and interceptors. This paradoxical situation arises since, using the double Lagrange multiplier method, there may

be many different values of λ and μ that correspond to the same resource levels. (See Reference 3 for a discussion of this point). In the preceding situation, there would be a larger value of λ that would correspond to the same total numbers of weapons and interceptors deployed.

The defense would prefer this latter deployment since it results in lower payoff to the attacker. This follows since, for $X_{min}(\lambda) \leq X \leq X_{max}(\lambda)$, the payoff to the attacker is the same per weapon (beyond the payoff of the "$X_{min}(\lambda)$ weapons"):

$$\text{Payoff} = \lambda(X - X_0) \ .$$

For $\lambda_1 < \lambda_2$, the payoff $\lambda_1 X_{01}$ of the "$X_{min}(\lambda_1)$ weapons" is greater than the payoff $\lambda_2 X_{02}$ of the "$X_{min}(\lambda_2)$ weapons." Also, $\lambda_1 X < \lambda_2 X$. Hence

$$\text{Payoff for } \lambda_1 = \lambda_1(X - X_{01}) < \lambda_2(X - X_{02}) = \text{Payoff for } \lambda_2.$$

Hence the defender would choose the deployment corresponding to λ_2. Thus we see that the attacker can attack on the range $X_{min}(\lambda) \leq X \leq X_{max}(\lambda)$ only if the defense is "complete" (i.e., the defender defends at level y' in all Case 1 and Case 2 targets).

VII. Determination of Solutions Which Are Simultaneously Optimal for Both Attacker and Defender

We shall now determine which solutions are simultaneously optimal for <u>both</u> the attacker and the defender.

A. Correct Solution When $X_{min}(\lambda) \leq X \leq X_{max}(\lambda)$

In the preceding section, we showed that the pair (μ,λ) corresponding to the correct solution cannot correspond to a partial defense and a partial attack. Rather, the correct (μ,λ) must correspond either to a complete attack, partial defense, or to a complete defense, partial attack. We also provide that in the complete defense case, in which $X_{min}(\lambda) \leq X \leq X_{max}(\lambda)$, if two different choices for (μ,λ) correspond to the same resource levels, the "better" solution corresponds to the larger value of λ. In view of the possibility of there being many different choices (μ,λ) that correspond to the same resource levels, it is necessary to answer the following question: How is it possible to find (μ,λ) corresponding to the solution optimal for the specified resource levels? Our objective is twofold: (1) to make certain that the correct (μ,λ) pair is included among all (μ,λ) pairs identified as corresponding to the specified resource levels; (2) to test each such (μ,λ) solution to determine if it is optimal.

In view of the results of the preceding section, if the correct solution is included in the set of (μ,λ) pairs identified as meeting the constraint levels, then we will be able to select it in the complete defense case. Hence our problem is solved for the complete defense case if we can prove that the correct solution is among those identified.

We can prove the correct solution is in fact included, by the following argument. We will show that there is only one permissible choice of λ corresponding to the complete defense case. Since the correct solution must correspond to the complete defense case. Since the correct solution must correspond to some choice of (μ,λ) that meets the resource constraints, and this choice is the only permissible one, it must then be the correct solution. (There may indeed be other values of λ which satisfy the resource constraints, but they will correspond to partial defenses and attacks, and hence to spurious solutions.)

Note that for values of λ greater than max(i) (P_i,β_i), no weapons and no interceptors are allocated to any target. Furthermore, as λ is decreased to zero, the maximal number of weapons that are allocated increased <u>monotonically</u> (but discontinuously) from zero. For $\mu > \lambda$, no interceptors are allocated, and for all $\mu \leq \lambda$, the allocation is the same (all $Y_{max}(\lambda)$ interceptors). The upper interceptor limit increases <u>monotonically</u> (and continously) as λ decreases. Hence, if we are allocating for a complete defense, partial attack, there is only <u>one</u> value of λ which will meet the defense level constraint. Hence we have the correct solution (i.e., the solution which is simultaneously optimal for both attacker and defender) in the complete defense case ($X_{min}(\lambda) \leq X \leq X_{max}(\lambda)$).

B. Correct Solution When $X < X_{min}(\lambda)$.

Note that there may be <u>no choice</u> of (μ,λ) corresponding to the specified defense and attack levels. For example, in the complete defense, minimal attack situation, the minimal attack resource level may exceed the specified attack resource level. In this case, we have no solution to the problem for the specified resource levels, by the Everett-Pugh <u>double</u> Lagrange multiplier technique. It is nevertheless possible to determine an optimal solution in this case using the <u>single</u> Lagrange multiplier technique, by the following reasoning. Consider the complete defense, minimal attack situation, in which the defender defends all Case 1 and Case 2 targets, and the attacker attacks all such targets at the minimal level (x_1'). The attacker deploys $X_{min}(\lambda)$ weapons in this case. If the attacker wishes to use fewer than $X_{min}(\lambda)$ weapons, he must allocate his weapons so that the marginal payoff per weapon is the same for all targets. That is, he chooses a new value of λ, greater than λ', and chooses x' so that $g'(x') = \lambda'$. Clearly, he will attack all targets with a number of weapons that is less than x_1', and hence less than the number of interceptors. Since this is the case, the defender has no need to adjust his defense. He can maintain the same defense no matter how many weapons (less than $X_{min}(\lambda)$) the attacker allocates, since in every case there are fewer weapons than defenders. Additional interceptors have absolutely no effect on the payoff. The problem hence reduces to one of simply allocating the attacker's weapons, using g(x) for the payoff function. Since the total number $X(\lambda)$ of attackers allocated, given λ, is a monotone decreasing function of λ. this is easily accomplished. Note that the defense can rearrange his allocation any way he pleases, so long as $y \geq x$ on every target, and the resulting allocation is still optimal. Only $X(\lambda)$ of the interceptors are actually needed.

Thus we have the correct solution in the case in which $X < X_{min}(\lambda)$.

C. Correct Solution When $X > X_{max}(\lambda)$

The final case to consider is the complete attack case in which $X > X_{max}(\lambda)$. For a specified number of defended targets, both $Y(\lambda)$ and $X(Y(\lambda),\lambda)$ increase as λ decreases. However, there may be several different numbers of defended targets for which $\lambda = \lambda'$ can be chosen so that $X(Y(\lambda'),\lambda') = X$ and $Y(\lambda')$ is close to Y. (Owing to the discrete nature of targets, it in general will not be possible to meet the interceptor constraint exactly.) In the unlikely event that $Y(\lambda')$ is the same for two different values of λ', the better solution corresponds to the allocation resulting in least damage. In general, however, we will have several possible solutions, each using a different number of interceptors and resulting in a different level of damage. The correct optimal solution is the solution with $Y(\lambda') \leq Y$ which has least damage. The situation is illustrated in Figure A-11.

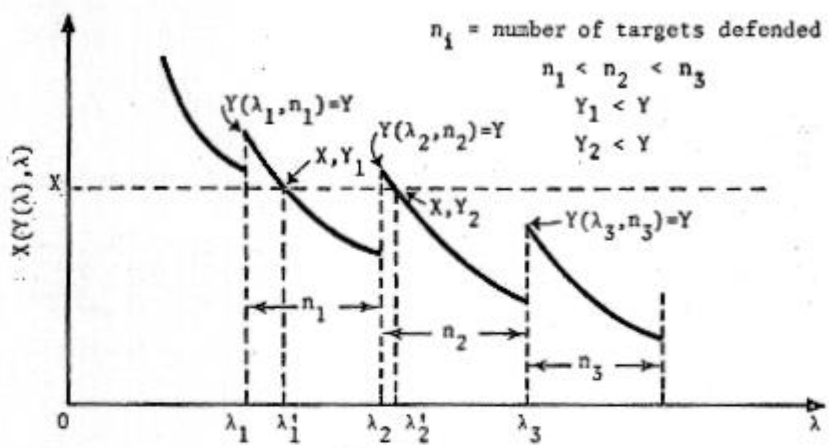

Figure A-11. Relationship of Total Number of Weapons Allocated $(X(Y(\lambda),\lambda))$ to Lagrange Multiplier (λ)

For both Case 1 and Case 2 targets, the damage in the optimal attack (complete attack case) is a function of λ only, i.e.,

$$\text{Damage} = f(x - y,y) = f(\lambda) = P(1 - \lambda/P\beta).$$

We observe that the damage is a decreasing function of λ. Therefore, if there are several different λ's for which the weapon constraint is met and for which the interceptor constraint is not exceeded, then the defender will choose the defense corresponding to the largest λ. Referring to Figure A-11, the optimal solution is hence the rightmost one (corresponding to total resource levels X, Y_2), in which the maximum number of targets is defended.

D. Summary of Correct Solutions

A useful way to summarize the several preceding observations is to plot the payoff for optimal solutions corresponding to varying attack levels for a fixed defense level. Such a plot is shown in Figure A-12.

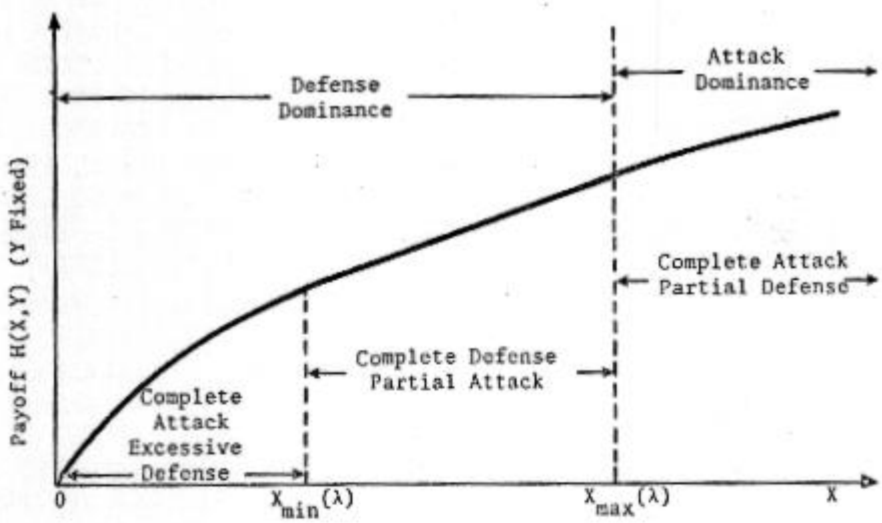

Figure A-12. Nature of Optimal Solution as Function of Attack Level, X. (Defense Level, Y, is Fixed)

Before summarizing the nature of the solutions corresponding to the different sections of this curve, we recall the nature of the Lagrangian solutions on the three types of targets.

For Case 1 targets, the defender chooses a number, y, of interceptors and the attacker chooses a number, x, of weapons, such that the tangent line to the damage curve at x is of slope λ, and the tangent line passes through the origin. (See Figure A-8.) For Case 2 targets, the defender chooses y, and the attacker chooses $x_2 > y$, so that the tangent line to the curve at x_2 is also tangent at a point $x_1 < y$; furthermore, this tangent line is of slope λ. (See Figure A-9.) Case 3 targets are neither defended nor attacked. (See Figure A-10.)

Not all of the points described above correspond to optimal solutions. Depending on the circumstances (to be described) the attacker may be required to attack only at the higher level, or the defender may be required to defend only at the higher level. We shall now describe the situation using Figure A-12, which depicts total damage, H(X,Y) as a function of X, for fixed Y.

In the first section of the curve, all Case 1 and Case 2 targets are defended, and all are attacked. The defense is greater than necessary in this region, for there are more interceptors than weapons on each target. The defense is somewhat arbitrary here; it can be rearranged in any fashion so long as there are always more interceptors than weapons on each target. The attack corresponding to a specified X is unique. Both the interceptor and weapon constraints are met exactly.

In the second section of the curve, all Case 1 and Case 2 targets are defended, and the defense is unique (or "stable").

The attacker may attack each target at either the low level or the high level, and he does so in any fashion which enables him to meet his weapon constraint, X. The interceptor constraint, Y, is met exactly, but the weapon constraint may only be approximated, owing to the discrete nature of weapons and interceptors. (We refer to this error as the "small change" effect.) Thus there are actually only a finite number of Lagrangian solutions along the linear portion of the curve corresponding to all possible combinations of attacks at the high or low attack levels on each target. Payoffs corresponding to total attack levels lying between these points lie slightly below the line defined by the payoffs of the Lagrangian solutions.

The first two sections of the curve represent what is called a "defense dominant" situation.

In the third section of the curve, all Case 1 and Case 2 targets are attacked at the higher level, but only a subset of the Case 2 targets are defended. The defended targets have the following property. Let λ denote the value of the (attacker's) Lagrange multiplier in the optimal attack. Suppose, now, that no targets were defended, and that this same value of λ were used to determine the number of weapons to be allocated to each target. Let us denote the damage to the i-th target in such an attack by D_i, and the number of weapons by N_i. Consider the ratio $R_i = D_i/N_i$. If all Case 1 and Case 2 targets are arranged in decreasing order of R_i, then the defended targets of the optimal attack are at the beginning of this list. In other words, the defender defends targets in decreasing order of their undefended damage per weapon.

Both the defense and the attack are unique in this section of the curve. The weapon constraint is met exactly, but the interceptor constraint may only be approximated. This section of the curve represents the "attack dominant" situation.

In the third section of the curve, there may be several different values of λ, corresponding to different numbers of defended targets, which enable us to meet the weapon and interceptor constraints. The defender naturally chooses the defense allocation which corresponds to the least damage. In the present problem, this happens to the solution corresponding to the largest value of λ, since in the attack dominant case the damage on the i-th target is given by

$$P_i(1 - \lambda/P_i\beta_i) .$$

VIII. Nature of the Solution as a Function of the Attacker's Lagrange Multiplier

It is of some interest to describe the nature of the solution as a function of the Lagrange multiplier λ. We summarize the situation in Figure A-13 through A-17 (ignoring the "small-change" effect caused by discrete weapons and interceptors).

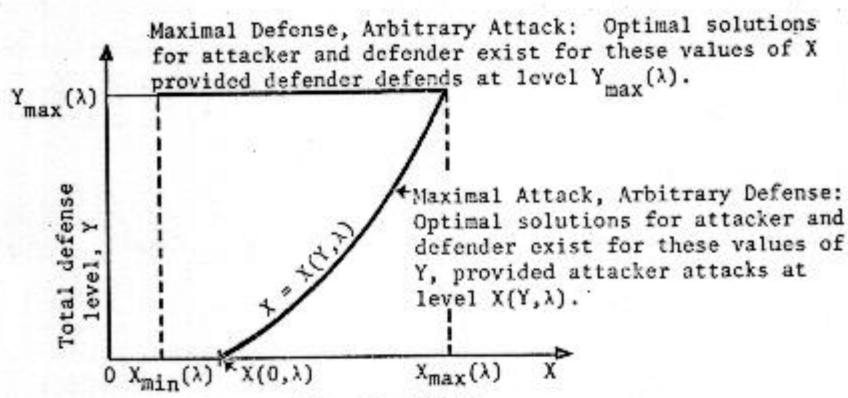

Figure A-13. Total Defense and Attack Levels Corresponding to Optimal Solutions (λ Fixed)

Figures A-14 and A-15 illustrate the ranges of total defense and attack levels corresponding to optimal attacks for different values of λ.

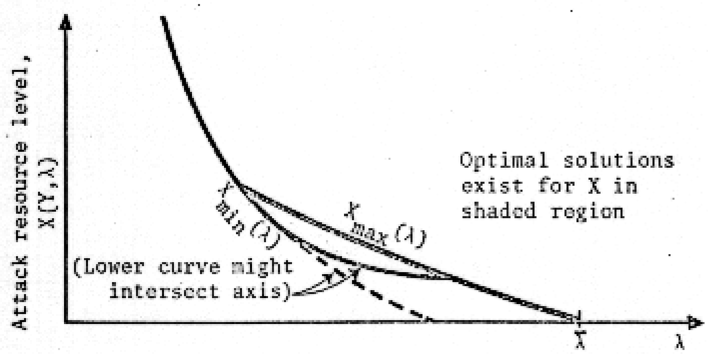

Figure A-14. Possible Optimal Attack Resource Levels for $Y = Y_{max}(\lambda)$

Note that it is not possible to obtain an optimal solution for every choice of X and Y, by the method of double Lagrange multipliers. If Y is very large and X is very small, there may be no value of λ for which $X_{min}(\lambda) < X$ and $Y < Y_{max}(\lambda)$. (This region is called a "gap". Regions such as this, which are inaccessible by the Everett-Pugh procedure of min-maxing the Lagrangian function (as opposed to zeroing its derivative, in the case of continuously differentiable functions), represent "inefficient" solutions. (See Reference 2 for a discussion of gaps.) As discussed previously, however, we were able in the present problem to find an optimal solution over this gap. (Another type of gap occurs since the discrete nature of targets prohibits meeting both resource levels exactly. This "small change" effect is generally small, however, for problems involving several targets and large numbers of weapons and interceptors.)

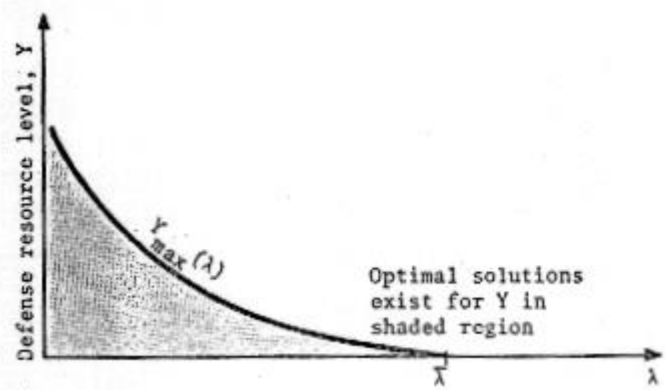

Figure A-15. Possible Optimal Defense Resource Levels for $X = X(Y,\lambda)$

Figure A-16 is a plot of the payoff (denoted by $H(X,Y_{max}(\lambda))$) corresponding to all possible optimal attack levels, for constant λ and $Y = Y_{max}(\lambda)$. (This plot is simply the middle section of Figure A-12.)

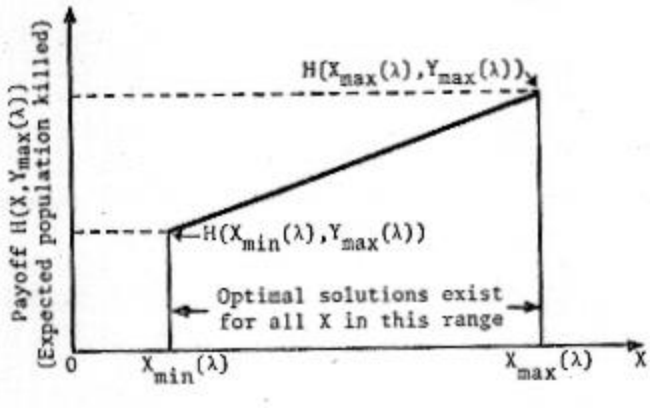

Figure A-16. Payoffs for Optimal Attacks (Complete Defense $Y_{max}(\lambda)$, Arbitrary Attack $X_{min}(\lambda) \leq X \leq X_{max}(\lambda)$) ($\lambda$ Fixed)

Figure A-17 is a plot of the payoff (denoted by $H(X(Y,\lambda),Y)$) corresponding to all possible optimal defense levels, for fixed λ.

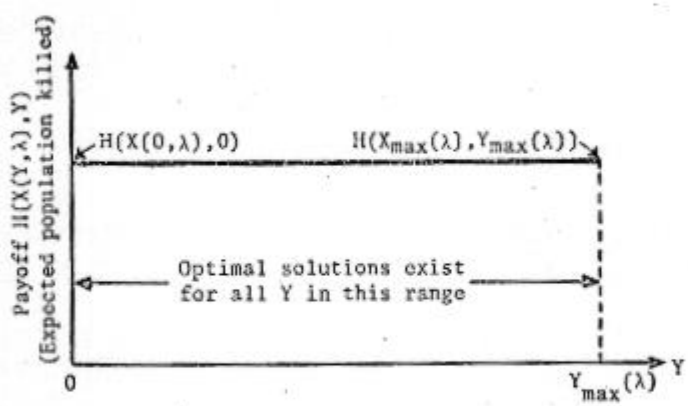

Figure A-17. Payoffs for Optimal Defenses (Complete Attack $X(Y,\lambda)$, Arbitrary Defense $0 \leq Y \leq Y_{max}(\lambda)$) ($\lambda$ Fixed)

The plot of Figure A-17 is a level line since $f(\Delta_\lambda,y) = (\Delta_\lambda,0)$, as discussed previously.

IX. Computer Program for Computing the Optimal Solution

In view of the preceding characteristics of the optimal solution, a reasonable procedure for finding the optimal solution (corresponding to $\Sigma\, x_i = X$, $\Sigma\, y_i = Y$) is the following:

1. Determine the largest value of λ such that $Y_{max}(\lambda) = Y$. Compute $X_{min}(\lambda)$ and $X_{max}(\lambda)$.

2. Complete defense, partial attack case. If $X_{min}(\lambda) \leq X \leq X_{max}(\lambda)$, then the optimal solution is obtained by attacking a sufficient number of the (Case 1 or Case 2) targets at the higher level (x_{2i}') to satisfy the weapon constraint, X. (Any such collection of targets is satisfactory. This is the case since the additional damage caused in going from x_{1i}' to x_{2i}' weapons on the i-th target is $\lambda(x_{2i}' - x_{1i}')$. Thus the additional damage caused by the $x_{2i}' - x_{1i}'$ weapons, above the damage caused by the x_{1i}' weapons, is the same linear function of the additional weapons allocated, for all targets.) Since weapons are allocated to targets in groups, it will in general not be possible to meet the weapon constraint exactly.

3. Complete attack, partial defense case. If $X > X_{max}(\lambda)$, then decrease λ to the largest value of λ such that $X_{max}(\lambda) = X$. An optimal solution is obtained by defending a sufficient number of the (Case 1 or Case 2) targets to satisfy the interceptor constraint, Y. The targets to be defended must be selected in order of decreasing ratio in the no-defense case, of damage to

weapons, until Y interceptors are used. Unfortunately, however, if some defendable targets are not defended, then fewer weapons are allocated (see Figure A-11). Hence our weapon constraint is not met. We must hence decrease the value of λ (giving us new values of $Y_{max}(\lambda) > Y$ and $X_{max}(\lambda) > X$), and choose a new set of targets to be defended. As before, we select targets to be defended in the (new) order of decreasing ratio, in the no-defense case, of damage to weapons, until Y interceptors are used. We compute, for this defense allocation, the corresponding number, X(Y), of weapons. We decrease λ until X(Y) = X. Since interceptors are allocated to targets in groups, it will in general not be possible to meet the interceptor constraint exactly. There may, however, be a number of different solutions (corresponding to different numbers of defended targets) using fewer than Y interceptors, and we should choose the one resulting in least damage (largest λ).

4. Complete attack, unnecessarily large defense. If $X < X_{min}(\lambda)$, no solution can be found by the method of double Lagrange multipliers. Either fewer than Y interceptors must be used, or more than X weapons must be used, to obtain an optimal solution by this method. To find the optimal solution for the specified Y, we instead use the single Lagrange multiplier method, and proceed as follows. Accept the defense allocation corresponding to $X_{min}(\lambda)$, $Y_{max}(\lambda) = Y$. Then, increase the value of λ, and compute the corresponding attack allocation. Denoting the number of weapons deployed by X(λ), adjust λ so that X = X(λ). This allocation is the optimal attack allocation. Denoting the number of weapons deployed by X(λ), adjust λ so that X = X(λ).

This allocation is the optimal attack allocation. Any defense allocation for which the number of interceptors on each target exceeds the number of weapons is optimal. Hence, in particular, the defense allocation corresponding to $X = X_{min}(\lambda)$, $Y = Y_{max}(\lambda)$, is optimal.

A computer program has been written to perform the above computations.

X. Explicit Solution

The derivation presented above of the optimal solution includes a number of implicit relationships. The solution will now be explicitly derived.

Case 3.

If $f'(0,0) \leq \lambda$, no weapons or interceptors are allocated to the target.

We have

$$f(\Delta,y) = P(1 - \exp(-\beta\Delta - \alpha y))$$

so that

$$f'(\Delta,y) = P\beta\exp(-\beta\Delta - \alpha y)$$

and

$$f'(0,0) = P\beta\ .$$

Thus if $P\beta \leq \lambda$, no weapons or interceptors are allocated to the target.

Case 1.

In this case, $P\beta > \lambda$ and $g'(0) \leq \lambda$. We have

$$g(x) = P(1 - \exp(-\alpha x))$$

so that

$$g'(x) = P\alpha\exp(-\alpha x)$$

and

$$g'(0) = P\alpha .$$

Thus if $P\alpha \leq \lambda$, the situation is that of Case 1. In this case, we wish first to determine $x,y > 0$ such that the tangent line at the point x is of slope λ and passes through the origin. That is, find x,y such that

$$f'(x - y, y) = \lambda$$

and

$$f(x - y,y)/x = \lambda .$$

The first relation implies

$$P\beta\exp(-\beta(x - y) - \alpha y) = \lambda$$

or

$$\exp(-\beta(x - y) - \alpha y) = \lambda/P\beta .$$

Substituting this into the second relation, we have

$$P(1 - \lambda/P\beta)/x = \lambda$$

or

$$x = P(1 - \lambda/P\beta)/\lambda .$$

The first relation then yields

$$y = (\ln(\lambda/P\beta) + \beta x)/(\beta - \alpha).$$

If $y = 0$, then x must satisfy

$$f'(x,0) = \lambda$$

i.e.,

$$P\beta\exp(-\beta x) = \lambda$$

or

$$x = (-1/\beta) \ln(\lambda/P\beta) .$$

Case 2

Case 2 corresponds to $P\alpha > \lambda$ ($P\beta >$ holds since $\alpha < \beta$). In this case, we wish first to determine x_1, x_2, and y, where $x_1 < y < x_2$, such that the tangent line at the point x_2 is of slope λ, the tangent line at the point x_1 is of slope λ, and these tangent lines are in fact the same line. That is, we want x_1, x_2, y such that

$$g'(x_1) = \lambda$$

$$f'(x_2 - y, y) = \lambda$$

$$(f(x_2 - y, y) - g(x_1))/(x_2 - x_1) = \lambda.$$

The first relation yields

$$\exp(-\alpha x_1) = \lambda/P\alpha$$

or

$$x_1 = -(1/\alpha)(\ln(\lambda/P\alpha)) .$$

The second relation yields

$$\exp(-\beta(x_2 - y))\exp(-\alpha y) = \lambda/P\beta .$$

Substituting the first two relations into the third hence yields

$$(P(1 - \lambda/P\beta) - P(1 - \lambda/P\alpha))/(x_2 - x_1) = \lambda$$

or

$$x_2 = x_1 + 1/\alpha - 1/\beta .$$

The second relation yields

$$y = (\ln(\lambda/P\beta) + \beta x_2)/(\beta - \alpha) .$$

In the case in which $y = 0$, we wish to find x such that

$$f'(x,0) = \lambda \ .$$

As is Case 1, we obtain

$$x = -(1/\beta)(\ln(\lambda/P\beta)) \ .$$

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FndKeywords(ballistic missile warfare; ballistic missile defense; resource-constrained game; min-max solution; generalized lagrange multipliers)